Chapter Six: Relations (and More)

6.1. Introduction: More Logical Form

1. Relations and Relation Letters. Once more we expand the formal language. And again our motivation is to ensure that the formal test of validity rightly evaluates intuitively valid arguments. The following argument, for example, strikes us as **valid**.[1]

1. Jack's a person, and he scaled the Cathedral of Learning.
2. The Cathedral of Learning is a skyscraper.

∴ Someone scaled a skyscraper.

The best translation we can manage in the Chapter Five language is the following.

a: Jack **b**: The Cathedral of Learning
G: __ is a person **H**: __ scaled the Cathedral of Learning
I: __ is a skyscraper **J**: __ scaled a skyscraper

1. (Ga ∧ Ha)
2. Ib

∴ ∃x (Gx ∧ Jx)

But this formal argument is susceptible to a simple validity counterexample.

[1] We've previously simplified our translation by leaving out the predicate "is a person" and restricting quantifier domain to just people (starting in 5.7). We bother to state the predicate "is a person" in the first premise just to allay suspicions that a tacit reference to people is causing the mismatch between English and formal arguments. Here, even with every predicate stated explicitly, the mismatch remains.

a: Jack **b**: The Cathedral of Learning
G: __ is a person **H**: __ scaled the Cathedral of Learning
I: __ is a skyscraper **J**: __ scaled a skyscraper

1. Jack's a person, and he scaled the Cathedral of Learning.	1. (Ga ∧ Ha)	**1**
2. The Cathedral of Learning is a skyscraper	2. Ib	**1**
∴ Someone scaled a skyscraper.	∴ ∃x (Gx ∧ Jx)	**0**

𝔻: {**Jack**, **The Cathedral of Learning**}

a: **Jack** **b**: **The Cathedral of Learning**
G: {**Jack**} **H**: {**Jack**}
I: {**The Cathedral of Learning**} **J**: { }

The formal test of validity judges this argument **invalid**.

Of course, consistent with our original intuition that the argument is valid, we're liable to balk at the situation presented as a counterexample: a case where, even though the Cathedral of Learning is a skyscraper, Jack scaled the Cathedral of Learning without scaling a skyscraper. Still, it would be a poor solution to rig the semantics to stamp this formal argument valid (the better to agree with our judgment of the English argument). For there are plenty of glaringly **invalid** arguments which take the same translation.

a: Neko **b**: Elvis
G: __ is a person **H**: __ ate fish tacos
I: __ is a gambler **J**: __ is a unicorn

1. Neko's a person, and she ate fish tacos.	1. (Ga ∧ Ha)
2. Elvis is a gambler.	2. Ib
∴ Someone is a unicorn.	∴ ∃x (Gx ∧ Jx)

a: Lucretia **b**: Letitia
G: __ is a person **H**: __ dyed his/her hair black
I: __ is feeling optimistic **J**: __ owns a magic wand

1. Lucretia's a person, and she dyed her hair black. 2. Letitia is feeling optimistic. --- ∴ Someone owns a magic wand.	1. (Ga ∧ Ha) 2. Ib --- ∴ ∃x (Gx ∧ Jx)

The problem with our formal translation of the original English argument is a familiar one: there are significant, but neglected, overlaps among the premises and conclusion of that argument. For example, the word "skyscraper" appears in both the second premise and the conclusion; yet the formal translation reveals no such overlap.

1. Jack's a person, and he scaled the Cathedral of Learning. 2. The Cathedral of Learning is a **skyscraper**. --- ∴ Someone scaled a **skyscraper**.	1. (Ga ∧ Ha) 2. **I**b --- ∴ ∃x (Gx ∧ **J**x)

As the above invalid arguments show, as far as the formal language is concerned the predicate letters "I" and "J" can mean any old thing, however unrelated.

The shortcomings of the translation are ones we can't overcome within the Chapter Five formal language. We have, for instance, no way to bring out the common features of "is a skyscraper" and "scaled a skyscraper" using just predicate letters (and likewise between "The Cathedral of Learning" and "scaled The Cathedral of Learning").[2]

[2] Moreover, the predicate "scaled a skyscraper" papers over the tacit existential quantifier in "a skyscraper". "Jack scaled a skyscraper" says: there' some object, x, which is a skyscraper and which Jack scaled.

Central to both those failings, the translation overlooks the repeated word "scaled". Note that this doesn't behave like a predicate of the sort treated in Chapter Five, such as "___ is a person" or "___is a gambler". For those predicates have one 'blank' to fill with a name. Putting "Jack" in those blanks, for instance, yields the (true) complete sentence "Jack is a person" and the (false) sentence "Jack is a gambler".

By contrast, adding "Jack" to "scaled" yields "Jack scaled," which isn't a complete sentence.

Yet note: if we add a second name, "the Cathedral of Learning," we do get a complete sentence: "Jack scaled the Cathedral of Learning". Whereas a predicate such as "is a gambler" is **one** filled **blank** short of a sentence, "scaled" needs **two blanks** filled to yield a complete sentence.

Reserving the term "predicate" for our one-place sentence-makers, we call a **two-place** sentence-maker such as "scaled" a **relation phrase** of English.[3]

To accommodate two-place relation phrases, our formal language will need a counterpart not found in the language of Chapter Five. So besides our earlier one-place predicate letters we now introduce two-place **relation letters**. The same letters that counted as predicate letters will be pressed into service here – though making clear the two blanks to be filled by adding a **numerical superscript** "**2**". We likewise mark a letter as a (mere) predicate letter by adding a superscript "**1**". Our translation keys will then look like the following example.

> **G^1a**: a is a person
> **H^2ab**: a scaled b

(We mark the blanks with name letters instead of "__" because for the first time we have multiple blanks, which may be filled by different terms.)

[3] Using 'phrase' very loosely here, to mean a string of words. A two-place string such as "is the same age as" won't count as a **grammatical** phrase of English, in the sense of being a 'natural part' (a 'constituent') of an English sentence. Here again the grammars of English and the formal language diverge.

And just as the subject followed a predicate letter, both blank-filling terms follow a relation letter.

a: Jack **H^2ab**: a scaled b
b: The Cathedral of Learning

Jack scaled the Cathedral of Learning: **H^2ab**

2. Relation Letters Extended. Once we understand how the predicate letters of old can be extended to more than one place, we see as well that there's no need to stop at two-place relations. For English speaks naturally of **three-place relations**.

a: Tucumcari **b**: Pittsburgh **c**: San Diego
G^3abc: a is between b and c

Tucumcari is between Pittsburgh and San Diego.

G^3abc

(The "and" might tempt us to treat this sentence as a conjunction of two smaller claims. But we should resist that temptation: the claim that "Tucumcari is between Pittsburgh and San Diego" is **not** a conjunction of the two smaller nonsensical pseudo-sentences "Tucumcari is between Pittsburgh" and "Tucumcari is between San Diego".)

Further examples of three-place relations appear in the sentences "The director of the commercial **replaced** Kitty with Barbie," "Jake **sent** Jezebel a letter," and "Neko **prefers** Jack to Suki".

In fact, once we see that relation letters can have any (integral) number of places, a further possibility suggests itself: a **zero-place relation letter**. That would be a capital letter which qualifies as a complete formal sentence on its own, with **no terms** added to it. That describes a **sentence letter**. So we can **add capital letters P through Z to our relation letters** – adding a superscript "0" to get a sentence letter, such as "P^0" or "Q^0".

These changes yield the following construction rules for the expanded formal language of Chapter Six.

Chapter Six Construction Rules

Terms:

T1. Name letters are terms

T2. Variables are terms

Atomic Formulas:

A1. A relation letter with ***n*** many places, followed by ***n*** many terms, is an atomic formula.

Formulas:

1. Atomic formulas are formulas.
2. If ● is a formula, then ~● is a formula.
3. If ● and ▲ are formulas, then (● ∧ ▲) is a formula.
4. If ● and ▲ are formulas, then (● ∨ ▲) is a formula.
5. If ● and ▲ are formulas, then (● →▲) is a formula.
6. If ● and ▲ are formulas, then (● ↔▲) is a formula.
7. If ★ is a variable and ● is a formula, then ∃★ ● and ∀★ ● are both formulas.

Of course the successes of previous chapters show that this expanded language isn't always needed to demonstrate argument validity. And even when multi-place relations are discussed, the simpler methods of earlier chapters may suffice to show that an argument is valid.

The following is one extreme example.

1. Tucumcari is between Pittsburgh and San Diego.

∴ It's not the case that Tucumcari isn't between Pittsburgh and San Diego.

Even if we sweep the entire English premise under a sentence letter, the formal argument still comes out valid.

P^0: Tucumcari is between Pittsburgh and San Diego.

1. Tucumcari is between Pittsburgh and San Diego.

∴ It's not the case that Tucumcari isn't between Pittsburgh and San Diego.

1. P^0

∴ $\sim\sim P^0$

But if wielding the full Chapter Six language on this argument is logical overkill, the valid argument which began this discussion shows that sometimes nothing less than the expanded language of Chapter Six suffices to formally detect an argument's validity.